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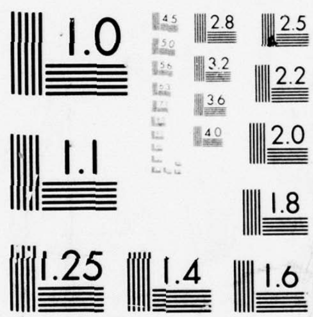


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# OPTIMAL GUIDANCE FOR MODULAR WEAPONS WITH DIGITAL AUTOPILOTS

by

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Prepared for

Department of the Air Force
Air Force Office of Scientific Research
Bolling Air Force Base, D.C. 20332

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# ABSTRACT

The optimal linear quadratic guidance problem with an accelerating target is investigated. The missile state is partitioned into a kinematic state and an airframe state. Both the penalty-weighted and constrained terminal state cases are treated. The resulting optimal guidance law requires estimates of target accelerations which are derived via linear observers. Results of a point mass missile, a one-time constant missile, and a two-time constant missile are given.

## ACKNOWLEDGMENT

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CONTENTS

## SECTION I

## INTRODUCTION

This report examines minimum control cost, minimum terminal miss guidance for the intercept of a maneuvering target by a missile with inherent airframe and autopilot dynamic properties. Particular attention is given to the idealized problem of zero terminal miss, wherein the control gains are given in terms of the state transition of the uncoupled airframe dynamics. This approach separates the kinematic portion of the intercept dynamics, which is common for all intercept problems, from the kinetic portion of the missile dynamics. The particular problem structure makes the results applicable to a variety of pursuit-evasion problems in which the missile airframe can be represented by a linear model.

The resulting control law developed in this paper has a term related to the intercept kinematics, which is recognizable as the generalized proportional navigation term. A second term in the control law is a linear function of the missile airframe state and represents the guidance compensation due to finite airframe dynamics. The final term in the guidance law is related to target motion, providing an effective control in cases where target motion can be measured or predicted accurately.

The next section outlines the problem. Subsequent sections develop a minimum miss, and a zero miss control for the general problem. A later section treats the estimation of the unknown target acceleration. Finally, the analysis is carried to conclusion for the case where the kinetics of the airframe are independent of the kinematic state.

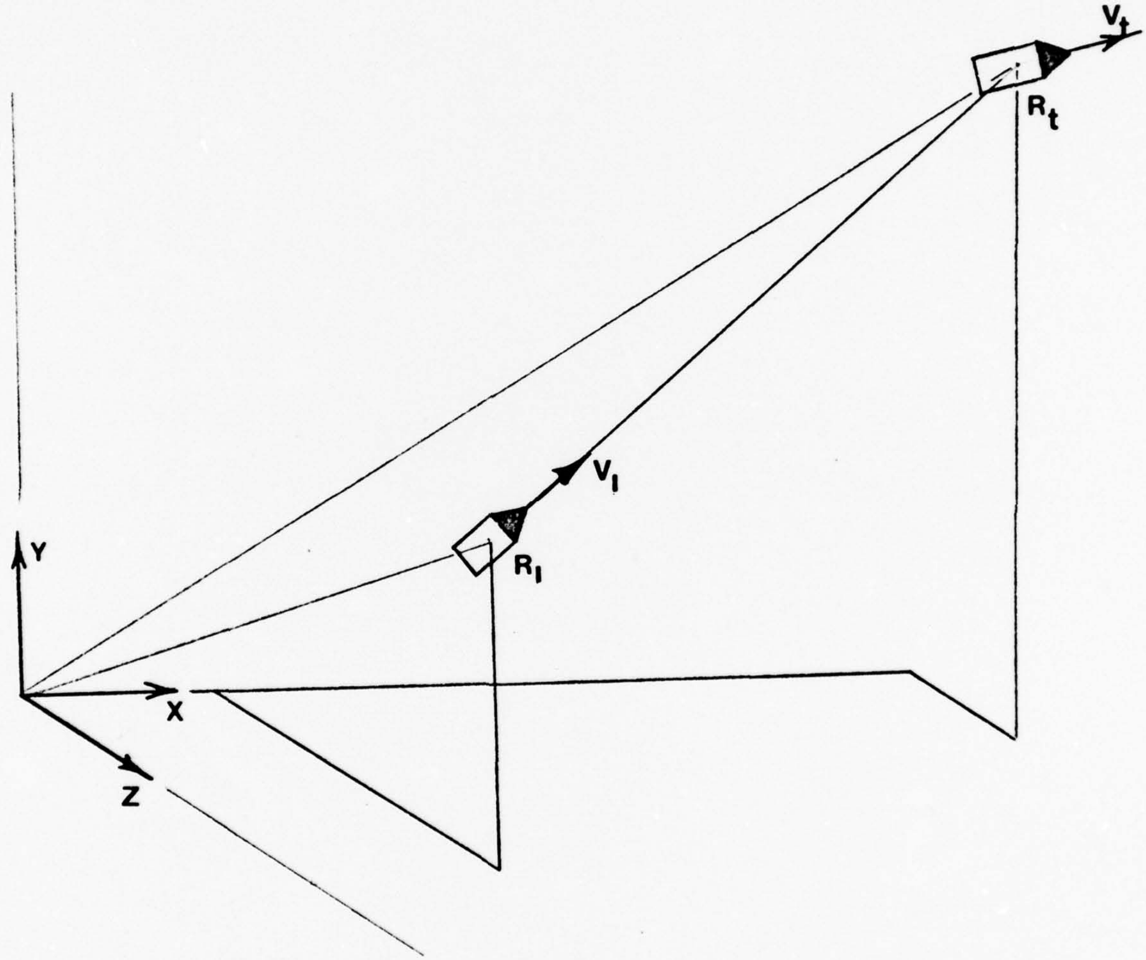


Figure 1. Engagement Geometry

## SECTION II

## FORMULATION OF THE GUIDANCE PROBLEM

This formulation structures the guidance problem to separate the kinematics of the intercept from the dynamics of the missile. The motion of the target is accepted as an uncontrollable input to the problem; however, the kinetic state equation can be augmented with a target model. Consider first the intercept kinematics.

### 1. KINEMATICS

Let the vector position and velocity of the target (T) and interceptor missile (I) be represented in an inertial frame by y and v, as illustrated in Figure 1.

$$\begin{aligned} \dot{y}_I &= v_I & \qquad \dot{y}_T &= v_T \\ \dot{v}_I &= a_I & \qquad \dot{v}_T &= a_T \end{aligned} \tag{1}$$

Defining the relative position and velocity of the target with respect to the missile yields

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= a_T - a_I \end{aligned} \tag{2}$$

Letting the vector x represent the kinematic state of the intercept,

$$x = \begin{bmatrix} y \\ v \end{bmatrix}$$

then

$$\dot{x} = Ax + B\,(a_I - a_T) \tag{3}$$

where

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \qquad B = -\begin{bmatrix} 0 \\ I \end{bmatrix}$$

In (3) the identity and null submatrices reflect the dimension of the problem.

## 2. KINETICS

The airframe/autopilot response state is designated as z and satisfies the linear equation

$$\dot{z} = Dx + Ez + Fu \tag{4}$$

subject to the airframe control u (thrust, control surface deflection, etc.). The dimensions and components of the coefficient matrices in (4) are missile dependent and provide the generality in the problem.

The intercept kinematics are coupled to the airframe dynamics by

$$a_I = Gz + Hu \tag{5}$$

Thus any linearized airframe describable by (4) and (5) is subject to the analysis.

## 3. OPTIMAL GUIDANCE

The conventional guidance performance index for homing missiles is of the form

$$J = \tfrac{1}{2}x(t_f)^T\, S_f x(t_f) + \int_{t_o}^{t_f} (\tfrac{1}{2}u^T\, Ru + \gamma)dt, \tag{6}$$

where $S_f$, R, and $\gamma$ weigh the costs associated with terminal miss, control cost and time respectively. In those cases where the terminal miss is the significant cost, it is logical to attempt to constrain the final position $y(t_f)$ to zero and to develop the corresponding control law under this condition. Equation (6) may be replaced by

$$J = \int_{t_o}^{t_f} (\tfrac{1}{2}u^T Ru + \gamma)dt$$

and (7)

$$Tx(t_f) = 0$$

where

$$T = \begin{bmatrix} I & 0 \end{bmatrix}$$

The equations (3-7) are collected below.

<u>Minimum Miss</u>

$$\dot{x} = Ax + BGz + BHu - Ba_T$$

$$\dot{z} = Dx + Ez + Fu \qquad (8)$$

$$J = \tfrac{1}{2}x(t_f)^T S_f x(t_f) + \int_{t_o}^{t_f} (\tfrac{1}{2}u^T Ru + \gamma)dt$$

<u>Zero Miss</u>

$$\dot{x} = Ax + BGz + BHu - Ba_T$$

$$\dot{z} = Dx + Ez + Fu$$

$$Tx(t_f) = 0 \qquad (9)$$

$$J = \int_{t_o}^{t_f} (\tfrac{1}{2}u^T Ru + \gamma)dt$$

## SECTION III

## MINIMUM TERMINAL MISS

If the performance index in (8) is augmented in the usual manner, the Hamiltonian is

$$H = \tfrac{1}{2}u^T Ru + \dot{\gamma} + \lambda^T [Ax + BGz + BHu - Ba_T] + \mu^T [Dx + Ez + Fu] \quad (10)$$

The resulting boundary value problem is

$$\begin{aligned}
\dot{x} &= Ax + BGz + BHu - Ba_T & x(t_o) &= x_o \\
\dot{z} &= Dx + Ez + Fu & z(t_o) &= z_o \\
\dot{\lambda} &= -A^T \lambda - D^T \mu & \lambda(t_f) &= S_f\, x(t_f) \\
\dot{\mu} &= -G^T B^T \lambda - E^T \mu & \mu(t_f) &= 0 \\
u &= -R^{-1} (H^T B^T \lambda + F^T \mu) & & \\
H(t_f) &= 0 & &
\end{aligned} \quad (11)$$

The computation of the control gains is achieved via the inverse formulation

$$x = Q_1 \lambda + Q_2 \mu + Q_3$$

$$z = Q_4 \lambda + Q_5 \mu + Q_6$$

leading to the equations

$$\begin{aligned}
\dot{Q}_1 &= AQ_1 + Q_1A^T + Q_2G^TB^T + BGQ_2{}^T - BHR^{-1}H^TB^T & Q_1(t_f) &= S_f{}^{-1} \\
\dot{Q}_2 &= AQ_2 + Q_2E^T + Q_1D^T + BGQ_5 - BHR^{-1} F^T & Q_2(t_f) &= 0 \\
\dot{Q}_3 &= AQ_3 + BGQ_6 - Ba_T & Q_3(t_f) &= 0
\end{aligned} \quad (12)$$

$$\dot{Q}_4 = Q_2^T$$

(12) (Contd.)

$$\dot{Q}_5 = EQ_5 + Q_5E^T + DQ_2 + Q_2^TD^T - FR^{-1}F^T \qquad Q_5(t_f) = S_f^{*-1}$$

$$\dot{Q}_6 = EQ_6 + DQ_3 \qquad Q_6(t_f) = 0$$

The nonsingular diagonal matrix $S_f^*$ is used for the computation of the inverse problem and its elements are set to zero in the solution for the gains.

The resulting control vector u is

$$u = K_1x + K_2z + K_3$$

where

$$K_1 = -R^{-1}(H^TB^TP_1 + F^TP_2^T)$$

$$K_2 = -R^{-1}(H^TB^TP_2 + F^TP_5) \tag{13}$$

$$K_3 = -R^{-1}(H^TB^TP_3 + F^TP_6)$$

and

$$P_1 = [Q_1 - Q_2Q_5^{-1}Q_2^T]^{-1}$$

$$P_2 = -[Q_1 - Q_2Q_5^{-1}Q_2^T]^{-1}Q_2Q_5^{-1}$$

$$P_5 = [Q_5 - Q_2^TQ_1^{-1}Q_2]^{-1} \tag{14}$$

$$P_3 = -P_1Q_3 - P_2Q_6$$

$$P_6 = -P_2^TQ_3 - P_5Q_6$$

The particular case of interest is that in which the kinetics of the airframe are independent of the intercept position and velocity, i.e.

$D = 0.$

In this case, the equations (12) are easily integrable yielding

$$
\begin{aligned}
Q_5 &= \int_t^{t_f} \Phi_E (t-\tau)\, FR^{-1} F^T \Phi_E{}^T (t-\tau)\, d\tau + S_f^{*-1} \\
Q_2 &= \int_t^{t_f} \Phi_A (t-\tau)\, B\, [HR^{-1} F^T - GQ_5 (\tau)]\, \phi_E{}^T(t-\tau)\, d\tau \\
Q_1 &= \int_t^{t_f} \phi_A (t-\tau)\, [BHR^{-1} H^T B^T - BGQ_2{}^T - Q_2 G^T B^T]\, \phi_A{}^T (t-\tau)\, d\tau + S_f{}^{-1} \\
Q_6 &= 0 \\
Q_3 &= -\int_t^{t_f} \phi_A (t-\tau)\, Ba_T (\tau) d\tau
\end{aligned}
\tag{15}
$$

where $\phi_A$ and $\phi_E$ are the transition matrices for A and E.

$$
\phi_A = \begin{bmatrix} I & tI \\ 0 & I \end{bmatrix}
$$

and

$$
\dot{\phi}_E = E\phi_E \quad , \quad \phi_E(0) = I.
$$

The elements of $S_f^*$ are set to zero in the resulting expressions for the elements of Q.

The minimum terminal miss control law is determined by (13) - (15). The case where the terminal miss weighting is large is solved by constraining the terminal position, as is done in the next section.

## SECTION IV

## ZERO TERMINAL MISS

The zero terminal miss, minimum control cost, minimum time problem formulated in an earlier section and outlined in (9) is solved here. This problem is also treated in reference 1. The augmented performance index for the problem

$$J = \nu^T Tx(t_f) + \int_{t_o}^{t_f} [\tfrac{1}{2}u^T Ru + \gamma + \lambda^T (Ax + BGz + BHu - Ba_T) + \mu^T (Dx + Ez + Fu)]dt$$

yields the boundary value problem described by

$$\begin{aligned}
\dot{x} &= Ax + BGz + BHu - Ba_T & x(t_o) &= x_o \\
\dot{z} &= Dx + Ez + Fu & z(t_o) &= z_o \\
\dot{\lambda} &= -A^T\lambda - D^T & \lambda(t_f) &= T^T\nu \\
\dot{\mu} &= -G^TB^T\lambda - E^T\mu & \mu(t_f) &= 0 \\
u &= -R^{-1}(H^TB^T\lambda + F^T\mu) & H(t_f) &= 0
\end{aligned} \qquad (16)$$

Selecting

$$\lambda = S_1x + S_2z + S_3\nu + S_4$$

$$\mu = S_5x + S_6z + S_7\nu + S_8$$

(16) becomes

$$\dot{S}_1 + A^T S_1 + S_1 A_2 + D^T S_5 + S_2 D_2 = 0 \qquad S_1(t_f) = 0$$

$$\dot{S}_2 + A^T S_2 + S_2 E_2 + D^T S_6 + S_1 B_2 = 0 \qquad S_2(t_f) = 0$$

$$\dot{S}_3 + A^T S_3 + D^T S_7 + S_1 C_2 + S_2 F_2 = 0 \qquad S_3(t_f) = T^T$$

$$\dot{S}_4 + A^T S_4 + S_2 H_2 + S_1 G_2 + D^T S_8 = 0 \qquad S_4(t_f) = 0$$

$$\dot{S}_5 + E^T S_5 + S_5 A_2 + G^T B_T S_1 + S_6 D_2 = 0 \qquad S_5(t_f) = 0 \tag{17}$$

$$\dot{S}_6 + E^T S_6 + S_6 E_2 + G^T B^T S_2 + S_5 B_2 = 0 \qquad S_6(t_f) = 0$$

$$\dot{S}_7 + E^T S_7 + G^T B^T S_3 + S_5 C_2 + S_6 F_2 = 0 \qquad S_7(t_f) = 0$$

$$\dot{S}_8 + E^T S_8 + G^T B^T S_4 + S_5 G_2 + S_6 H_2 = 0 \qquad S_8(t_f) = 0$$

where

$$A_2 = A - BHR^{-1} (H^T B^T S_1 + F^T S_5)$$

$$B_2 = BG - BHR^{-1} (H^T B^T S_2 + F^T S_6)$$

$$C_2 = -BHR^{-1} (H^T B^T S_3 + F^T S_7)$$

$$D_2 = D - FR^{-1} (H^T B^T S_1 + F^T S_5)$$

$$E_2 = E - FR^{-1} (H^T B^T S_2 + F^T S_6) \tag{18}$$

$$F_2 = -FR^{-1} (H^T B^T S_3 + F^T S_7)$$

$$G_2 = -BHR^{-1} (H^T B^T S_4 + F^T S_8) - Ba_T$$

$$H_2 = -FR^{-1} (H^T B^T S_4 + F^T S_8)$$

Inspection of (17) and (18) shows that all unknown matrices except $S_3$ and $S_7$ are null. Leaving

$$\dot{S}_3 + A^T S_3 + D^T S_7 = 0 \qquad S_3(t_f) = T^T$$

$$\dot{S}_7 + E^T S_7 + G^T B^T S_3 = 0 \qquad S_7(t_f) = 0 \tag{19}$$

$$u = -R^{-1} (H^T B^T S_3 + F^T S_7)\nu$$

The invariance of the terminal manifold

$$\Psi(t) = S_9 X + S_{10} Z + S_{11}\nu + S_{12}$$

implies

$$\dot{S}_9 + S_9 A + S_{10} D = 0 \qquad S_9(t_f) = T$$

$$\dot{S}_{10} + S_{10} E + S_9 BG = 0 \qquad S_{10}(t_f) = 0$$

$$\dot{S}_{11} - (S_9 BH + S_{10} F) R^{-1} (H^T B^T S_3 + F^T S_7) = 0 \qquad S_{11}(t_f) = 0 \tag{20}$$

$$\dot{S}_{12} - S_9 B a_T = 0 \qquad S_{12}(t_f) = 0$$

where

$$\nu = -S_{11}^{-1} [S_9 x + S_{10} z + S_{12}]$$

Therefore the control law for zero miss is

$$u = R^{-1} (H^T B^T S_3 + F^T S_7) S_{11}^{-1} (S_9 x + S_{10} z + S_{12}),$$

where the navigation, airframe, and target motion components are evident.

$$K_1 = R^{-1} (H^T B^T S_3 + F^T S_7) S_{11}^{-1} S_9$$

$$K_2 = R^{-1} (H^T B^T S_3 + F^T S_7) S_{11}^{-1} S_{10}$$

$$K_3 = R^{-1} (H^T B^T S_3 + F^T S_7) S_{11}^{-1} S_{12}$$

For the uncoupled airframe case where D = 0, the equations (19) and (20) are integrated for computation of the optimal gains. Integration yields

$$S_3 = S_9^T = \begin{bmatrix} I \\ \\ (t_f - t) I \end{bmatrix}$$

$$S_{10} = S_7^T = \int_t^{t_f} S_9(\tau) \, BG\Phi^*_E \, (t-\tau) \, d\tau$$

$$S_{11}(t) = - \int_t^{t_f} [S_{10}(\tau) F + S_9(\tau) BH] R^{-1} [S_{10}(\tau) F + S_9(\tau) BH]^T d\tau \tag{21}$$

$$S_{12}(t) = - \int_t^{t_f} S_9(\tau) \, Ba_T(\tau) \, d\tau$$

where $\dot{\Phi}^*_E = -\Phi^*_E E \qquad \Phi^* (0) = I$

The target maneuvering term of the control for the uncoupled case can be written

$$K_3 = K_{30} \int_t^{t_f} (t_f - \tau) \, a_T(\tau) \, d\tau \tag{22}$$

where $\quad K_{30} = R^{-1} (H^T B^T S_3 + F^T S_7) S_{11}^{-1}$

Also

$$K_3 = K_{30} (t_f - t) [\bar{V}_T(t) - V_T(t)] \tag{23}$$

where $V_T(t)$ represents present target velocity and $\bar{V}(t)$ is the average relative velocity on the terminal interval $(t, t_f)$. The need for effective estimation of target motion is recognized.

## SECTION V

## TARGET MANEUVERING

For minimum miss control the target maneuver term of the control is

$$u_T = -K_1Q_3 - K_2Q_6 \tag{24}$$

where

$$K_1 = -R^{-1}(H^TB^TP_1+F^TP_2{}^T)$$

$$K_2 = -R^{-1}(H^TB^TP_2+F^TP_5)$$

Also

$$Q_3 = \int_t^{t_f} \tilde{\Phi}_{33}(t-\tau)\ Ba_T(\tau)d\tau$$

$$Q_6 = \int_t^{t_f} \tilde{\Phi}_{63}(t-\tau)\ Ba_T(\tau)d\tau$$

where

$$\tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_{33} & \tilde{\Phi}_{36} \\ \tilde{\Phi}_{63} & \tilde{\Phi}_{66} \end{bmatrix}$$

is the state transition matrix corresponding to

$$\tilde{A} = \begin{bmatrix} A & BG \\ D & E \end{bmatrix}$$

For zero miss guidance the target maneuver component of the control reduces to

$$u_T = -R^{-1}(H^TB^TS_3+F^TS_7)\int_t^{t_f} S_9(\tau)Ba_T(\tau)d\tau. \tag{25}$$

In both the minimum miss and zero miss problems, the segment of the control due to the maneuvering target requires a weighted integral of $a_T$. This control can be implemented for the deterministic case if a reliable estimate of $a_T$ can be obtained from an on-line reconstructor. Such an estimate may be taken from a linear observer, if the unknown target acceleration is modeled by a linear equation

$$\dot{a}_T = M_1 a_T + M_2 \tag{26}$$

This assumption is partially justified by consideration of the dominant time constants of the target compared to the normal intercept time. In most cases a constant or ramp acceleration adequately describes the target motion in this interval. If the states x and z are completely measurable,

$$y = \begin{bmatrix} x \\ z \end{bmatrix}$$

a simple reduced-order linear observer for $a_T$ may be used. To facilitate the development, this approach is taken, although a higher order observer can be used if components of x and z are not directly measurable but are observable.

The equations are

$$\dot{x} = Ax + BGz + BHu - Ba_T$$

$$\dot{z} = Dx + Ez + Fu$$

$$\dot{a}_T = M_1 a_T + M_2$$

If an estimate $\hat{a}_T$ satisfies

$$\hat{a}_T = \tilde{a}_T + Lx \tag{27}$$

where

$$\dot{\tilde{a}}_T = (LB+M_1)\hat{a}_T - L(Ax+BGz+BHu) + M_2 \tag{28}$$

then

$$\dot{\hat{a}}_T - \dot{a}_T = (LB+M_1)(\hat{a}_T - a_T) \tag{29}$$

and the matrix L may be chosen to give the desired convergence for the observer. In the actual implementation, line-of-sight angles and rates will replace position and velocity in the observer model, and some components of z can be added to the observed state, as mentioned.

## SECTION VI

## TARGET INTERCEPT SITUATIONS
## USING
## OPTIMAL GUIDANCE WITH TARGET ESTIMATION

Situation I Kinematic Example

A preliminary example is one of a longitudinal planar intercept of a target with an unknown, constant acceleration by a point mass missile. To facilitate the solution, a zero miss constraint with quadratic missile acceleration cost is assumed. Since the lateral motion is completely uncoupled, the same type of control is applicable in that plane.

The equations are

$$\dot{y} = v$$

$$\dot{v} = a_T - a_I$$

$$J = \tfrac{1}{2} \int_o^{t_f} a_I^2 \, dt \quad , \quad y(t_f) = 0$$

Here

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$D = 0 \qquad H = 1 \qquad R = 1$$

Integration yields from (21)

$$S_9 = \begin{bmatrix} 0 & t_g \end{bmatrix}$$

$$S_{11} = -1/3 \; t_g^{\;3}$$

$$S_{12} = \int_t^{t_f} (t_f - \tau) \; a_T(\tau) d\tau$$

where $$t_g = t_f - t$$

The control is

$$a_I = (3/t_g^2)y + (3/t_g)v + (3/t_g^2)\int_t^{t_f}(t_f-\tau)\ a_T(\tau)d\tau$$

If the coordinate axes are chosen such that the down-range coordinate is along a non-rotating line of sight, the line-of-sight angle $\lambda$ is

$$\lambda \approx y(t)/r_g(t)$$

where $r_g(t)$ is the range-to-go, which depends on the target and missile accelerations, as well as present relative velocity and position.

$$y = \lambda r_g$$

$$v = \dot{\lambda} r_g + \lambda \dot{r}_g$$

Hence

$$a_I = (3/t_g^2)(r_g + \dot{r}_g t_g) + (3r_g/t_g)\dot{\lambda} + (3/t_g^2)\int_t^{t_f}(t_f-\tau)\ a_T(\tau)d\tau$$

If the range-to-go is decreasing steadily along the line of sight,

$$r_g = r_o t_g$$

the commanded acceleration becomes

$$a_I = 3r_o\dot{\lambda} + (3/t_g^2)\int_t^{t_f}(t_f-\tau)\ a_T(\tau)d\tau$$

In any case, if the control for intercept of a maneuvering target is required, the target acceleration normal to the line of sight $a_T$ must be estimated. If $a_T$ is an unknown constant as specified, it is modeled

$$\dot{a}_T = 0$$

Hence

$$M_1 = 0$$

$$M_2 = 0$$

In this case the target observer model is

$$\hat{a}_T = \tilde{a}_T + \ell_2 v$$

$$\dot{\tilde{a}}_T = -\ell_2 \hat{a}_T - \ell_2 a_I$$

where $\ell_2$ may be taken large enough to satisfy convergence requirements for the observer.

If a constant estimate of $a_T$ is achieved, then

$$a_I = (3/t_g^2)y + (3/t_g)v + 3/2\hat{a}_T$$

Due to the terminal constraint, the miss distance will be reduced to zero no matter what error exists in the target estimate (at, perhaps, a prohibitive terminal control gain). The control cost will, however, reflect the error in $\hat{a}_T$.

For comparison, assume that

$$\hat{a}_T = \gamma a_T$$

then

$$a_I = (3/t_g^2)y + (3/t_g)v + (3/2)\gamma a_T$$

Thus $\gamma = 0$ corresponds to ignoring target maneuvers, and $\gamma = 1$ corresponds to optimal control. The reduction in control cost from that obtained by not using a target estimate is

$$\Delta J = 9(2-\gamma)\gamma a_T^2 t_f \;/8$$

While any target estimate (for constant accelerating target) such that $0<\gamma<2$ represents an improvement in cost, the minimum cost is obviously achieved when the estimate is perfect.

## Situation II  One-Time-Constant Missile

A more realistic situation occurs when an airframe state exists in the missile model. In this case the equations of intercept are

$$\dot{y} = v$$

$$\dot{v} = a_T - a_I$$

$$\dot{z} = \alpha(u - z)$$

$$a_I = gz$$

Referring to (3, 4, 5)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$D = (0 \quad 0) \qquad E = -\alpha$$

$$F = \alpha \qquad G = g$$

$$H = 0$$

Setting

$$R = \lambda$$

and

$$s = 1$$

$$K_1 = -\frac{\alpha}{\lambda} P_2^T$$

$$K_2 = -\frac{\alpha}{\lambda} P_5$$

$$K_3 = \frac{\alpha}{\lambda} (P_2^T Q_3 + P_5 Q_6)$$

where

$$P_2 = We^{\alpha t_g} \begin{bmatrix} 1 \\ t_g \end{bmatrix}$$

$$P_5 = gWe^{\alpha t_g} \left(\frac{1}{\alpha^2} - \frac{t_g}{\alpha} - \frac{e^{-\alpha t_g}}{\alpha^2}\right)$$

$$Q_3^T = \left[ -\int_t^{t_f} (t-\tau)\ a_T(\tau) d\tau \qquad -\int_t^{t_f} a_T(\tau) d\tau \right]$$

$$Q_6 = 0$$

where

$$t_g = t_f - t$$

Finally,

$$W = \frac{6\alpha g\lambda e^{-\alpha t_g} (1 - e^{\alpha t_g} - \alpha t_g)}{6\alpha^3 g + 2\alpha^3 g^2 t_g^3 + 6\alpha g^2 t_g - 6\alpha^2 g^2 t_g - 12\alpha g^2 t_g e^{-\alpha t_g} + 3g^2 - 3g^2 e^{-2\alpha t_g}}$$

The substitution yields

$$u = K_{11} y + K_{12} v + K_2 z + K_3$$

where

$$K_{11} = -\frac{6\alpha^2 g (1 - e^{\alpha t_g} - \alpha t_g)}{6\alpha^3 \beta + 2\alpha^3 g^2 t_g^3 + 6\alpha g^2 t_g - 6\alpha^2 g^2 t_g^2 - 12\alpha g^2 t_g e^{-\alpha t_g} + 3g^2 - 3g^2 e^{-\alpha t_g}}$$

$$K_{12} = K_{11}\ t_g$$

$$K_2 = K_{11} \left(\alpha^{-2} - \alpha^{-1} t_g - \alpha^{-2} e^{-\alpha t_g}\right)$$

$$K_3 = K_{11} \int_t^{t_f} (t_f - \tau)\ a_T(\tau)\, d\tau$$

If a reduced observer is used to reconstruct the target acceleration for $K_3$, equations (27, 28) yield

$$\hat{a}_T = \tilde{a}_T + \ell_1 y + \ell_2 v$$

$$\dot{\tilde{a}}_T = (-\ell_2 + m_1)\hat{a}_T - \ell_1 v + \ell_2 gz + m_2$$

where the error $e = \hat{a}_T - a_T$ decays according to

$$\dot{e} = -\ell_2 e$$

If the target acceleration is almost constant over the duration of the intercept, $m_1$ and $m_2$ are taken as zero. If $\ell_1$ is taken to be zero for convenience,

$$\hat{a}_T = \tilde{a}_T + \ell_2 v$$

$$\dot{\tilde{a}}_T = -\ell_2 \hat{a}_T + \ell_2 gz$$

The observer is shown in Figure 2.

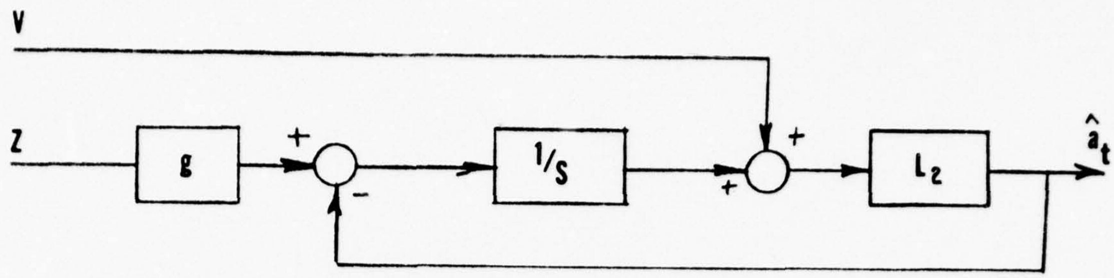


Figure 2. Linear Observer

Typical intercepts are shown in Figures 4 - 12. Parameter values for these simulations were chosen as follows:

$$\alpha = 1$$

$$\beta = 1$$

$$g = 1$$

$$t_f = 2$$

Situation III Two-Time-Constant Missile

A still more realistic tactical missile situation exists when the rigid-body rotation state is included, as described in Figure 3.

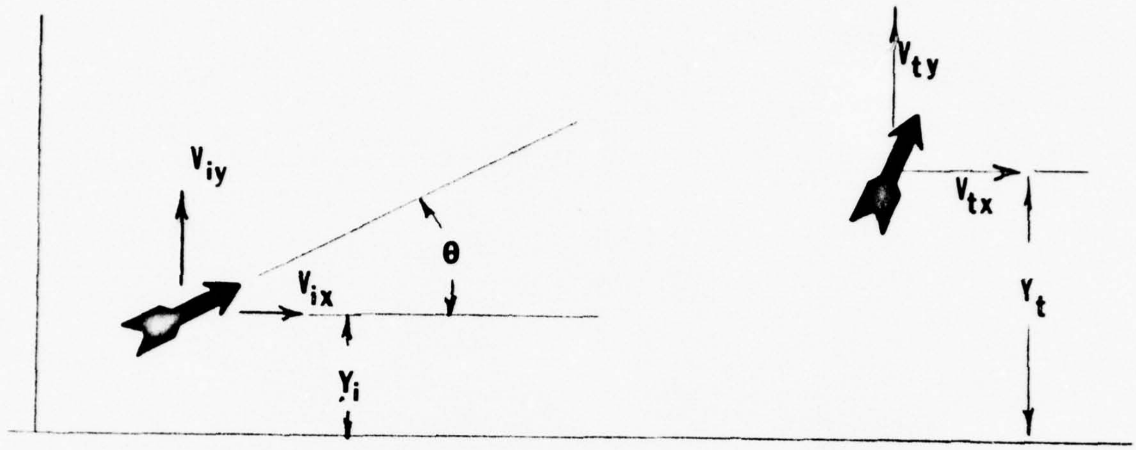


Figure 3. Two-Time-Constant Missile

If the initial line of sight is taken as the x-axis, for small θ the equations of intercept are as follows:

$$y_T - y_I = y$$

$$v_{Tx} - v_{Ix} = -v_o \quad , \quad \dot{v}_o = 0$$

$$v_{Ty} - v_{Iy} = v$$

$$\dot{v}_{Iy} = (T/M)\theta$$

$$\ddot{\theta} = \left(\frac{C_1}{I}\right)\dot{\theta} + \left(\frac{C_2}{I}\right)u$$

yielding

$$\dot{y} = v$$

$$\dot{v} = g\theta + a_T$$

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \alpha\omega + \beta u$$

An optimal guidance problem is formulated with

$$J = y(t_f)^2 + \lambda \int_o^{t_f} u^2 dt$$

In the format of Section II,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad F = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad G = (-g \quad 0)$$

$$H = 0$$

Solving for the Q and subsequent gain matrices in (13-15)

$$K_1(t_g) = K_{11}(t_g) \; [1 \qquad t_g]$$

$$K_2(t_g) = K_{11}(t_g) \; [- \frac{g{t_g}^2}{2} - W(t_g)e^{-\alpha t_g}]$$

$$K_3(t_g) = K_{11}(t_g) \int_t^{t_f} (t_f - \tau) \; a_T(\tau) d\tau$$

where

$$K_{11}(t_g) = \frac{\beta W(t_g)}{\lambda V(t_g)}$$

$$W(t_g) = \frac{g}{\beta^3}[(\tfrac{1}{2}\alpha^2 t_g{}^2 - \alpha t_g + 1)e^{\alpha t_g} - 1]$$

and

$$V(t_g) = e^{\alpha t_g}[C_1(t_g) + t_g C_2(t_g) - \frac{g t_g{}^2}{2} C_3(t_g)]$$

$$- W(t_g)\, C_4(t_g)$$

$$C_1(t_g) = 1 + \frac{g^2\beta^2}{\lambda\alpha^7}[\frac{\alpha^5 t_g{}^5}{120} + \frac{\alpha^3 t_g{}^3}{6} + \alpha t_g - \sinh(\alpha t_g)]$$

$$C_2(t_g) = \frac{g\beta^3}{\lambda\alpha^6}[-\frac{\alpha^4 t_g{}^4}{24} - \frac{\alpha^2 t_g{}^2}{2} - 1 + \cosh(\alpha t_g)]$$

$$C_3(t_g) = \frac{g\beta^2}{\lambda\alpha^5}[-\frac{\alpha^3 t_g{}^3}{6} - \alpha t_g + \sinh(\alpha t_g)]$$

$$C_4(t_g) = \frac{g\beta^2}{\lambda\alpha^4}[-\frac{\alpha^2 t_g{}^2}{2} + 1 - \cosh(\alpha t_g)]$$

The optimal control deflection is

$$u_{opt} = K_{11}y + K_{12}v + K_{21}\theta + K_{22}\omega + K_3$$

Figures 13-20 give typical trajectories for this intercept.

Parameter values for these simulations were taken as follows:

$$\alpha = 1$$

$$\beta = 1$$

$$g = 100$$

$$t_f = 2$$

$$\lambda = 1$$

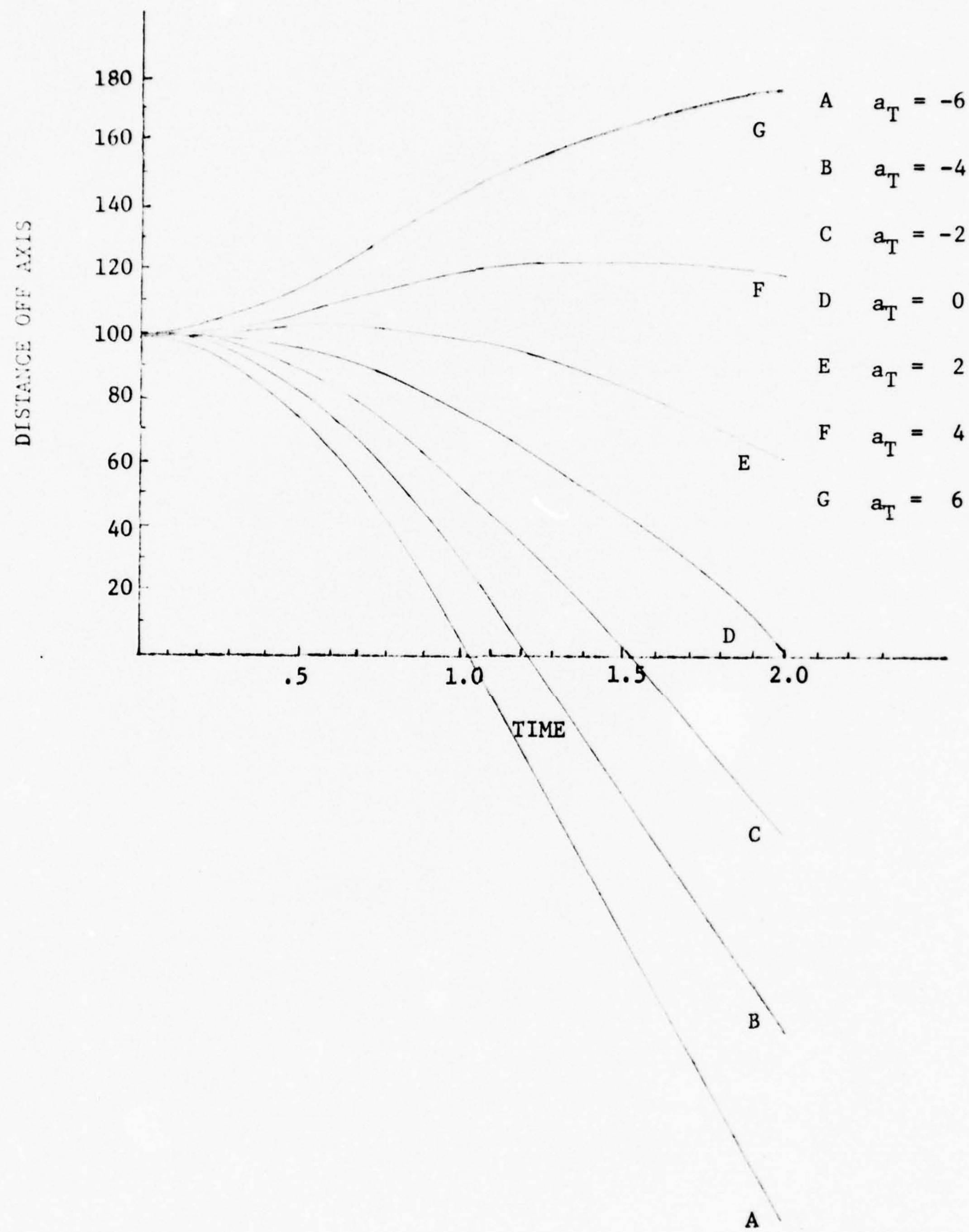


Figure 4. Miss Distance for One-Time Constant Missile Without Target Estimation

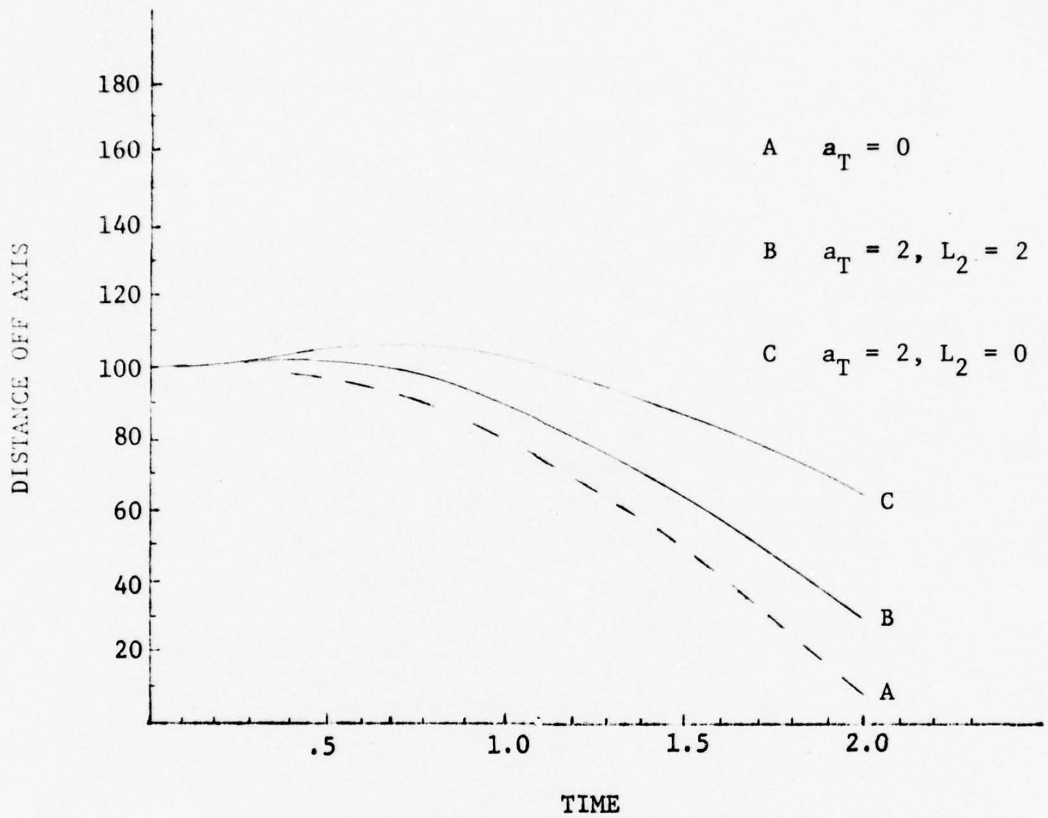


Figure 5. Miss Distance for One-Time Constant Missile.
Target Acceleration = 2g

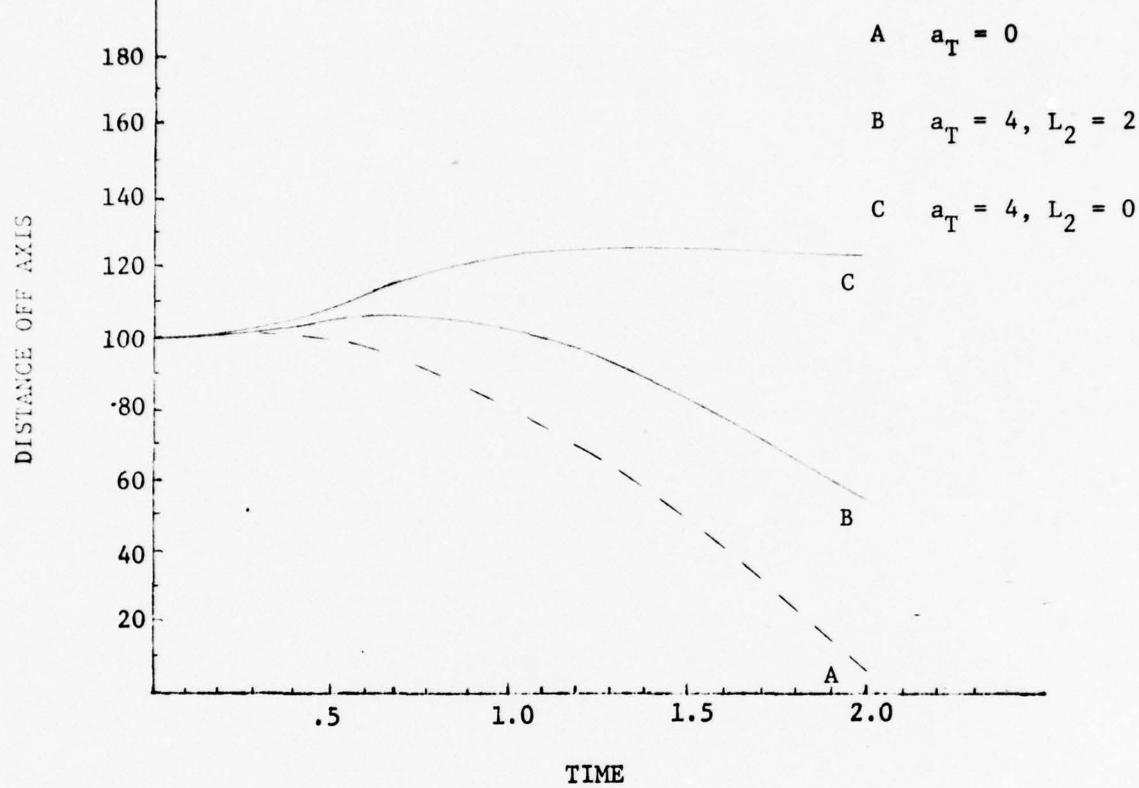


Figure 6. Miss Distance for One-Time Constant Missile.
Target Acceleration = 4g

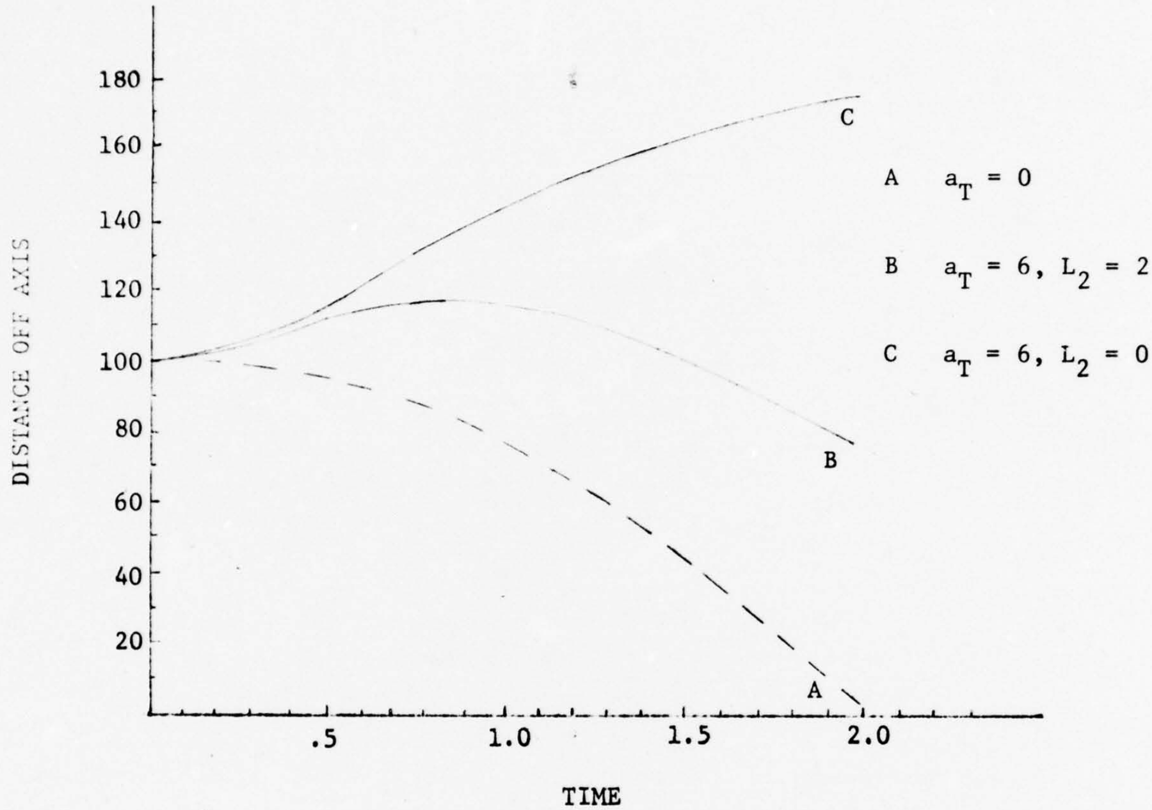


Figure 7. Miss Distance for One-Time Constant Missile.
Target Acceleration = 6g

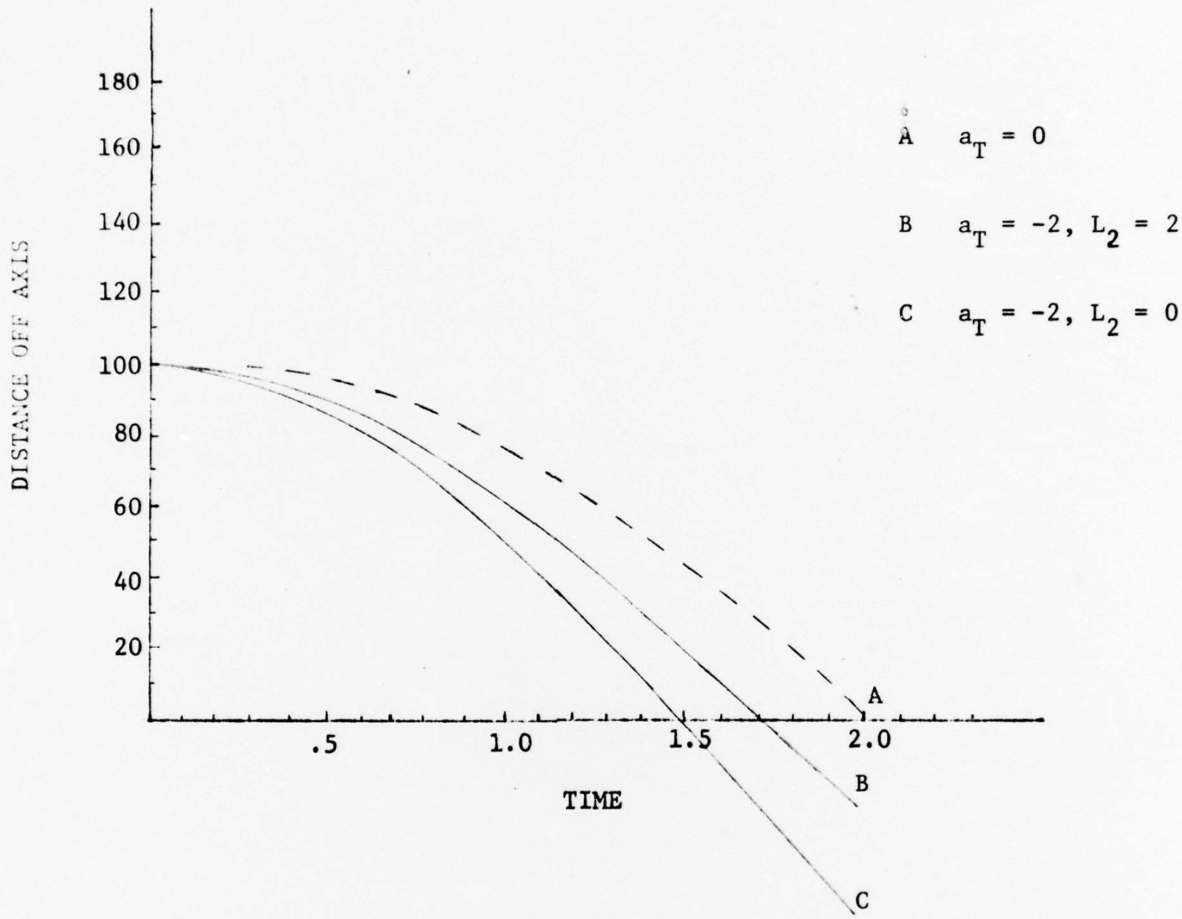


Figure 8. Miss Distance for One-Time Constant Missile. Target Acceleration = -2g

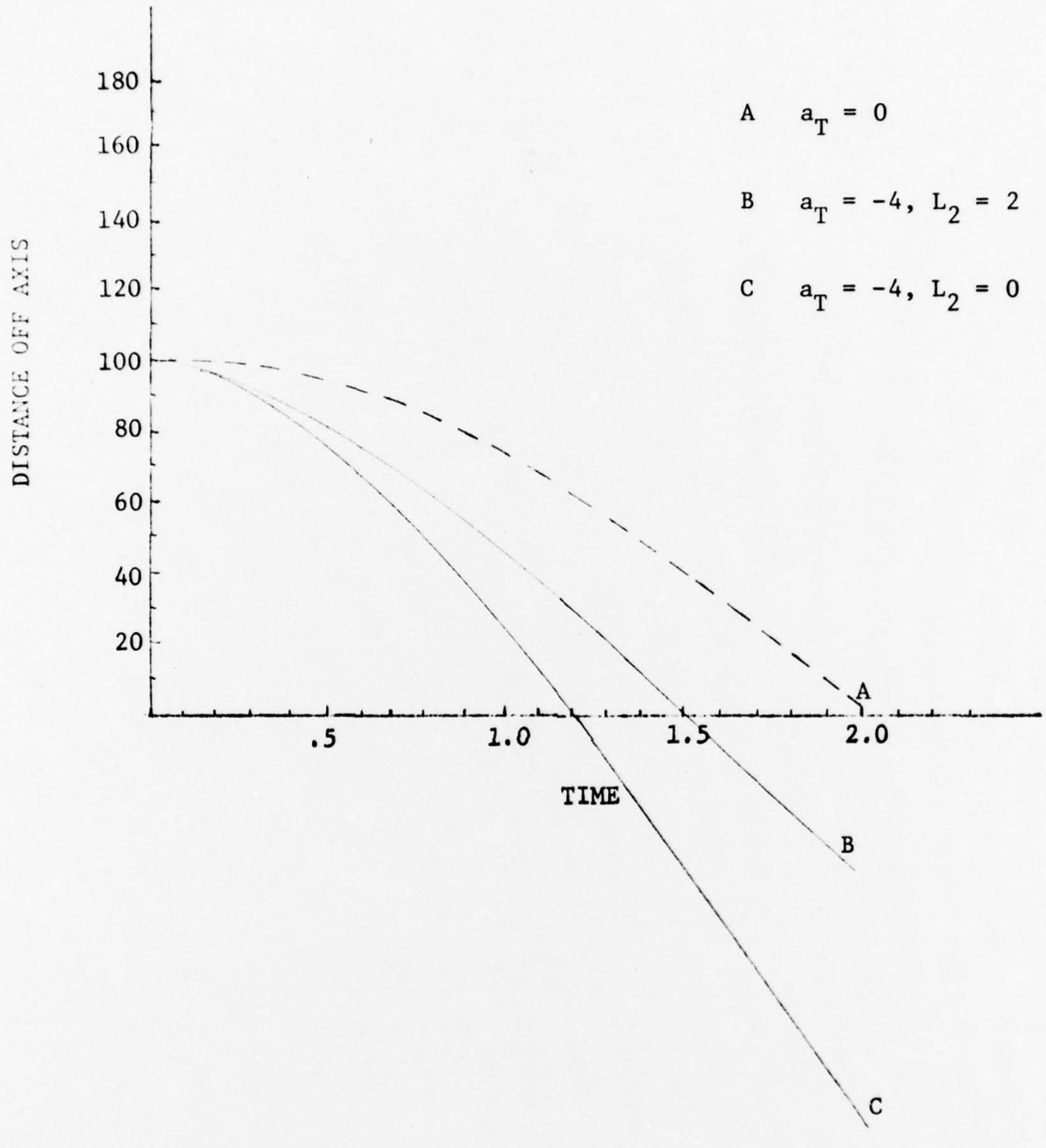


Figure 9. Miss Distance for One-Time Constant Missile.
Target Acceleration = -4g

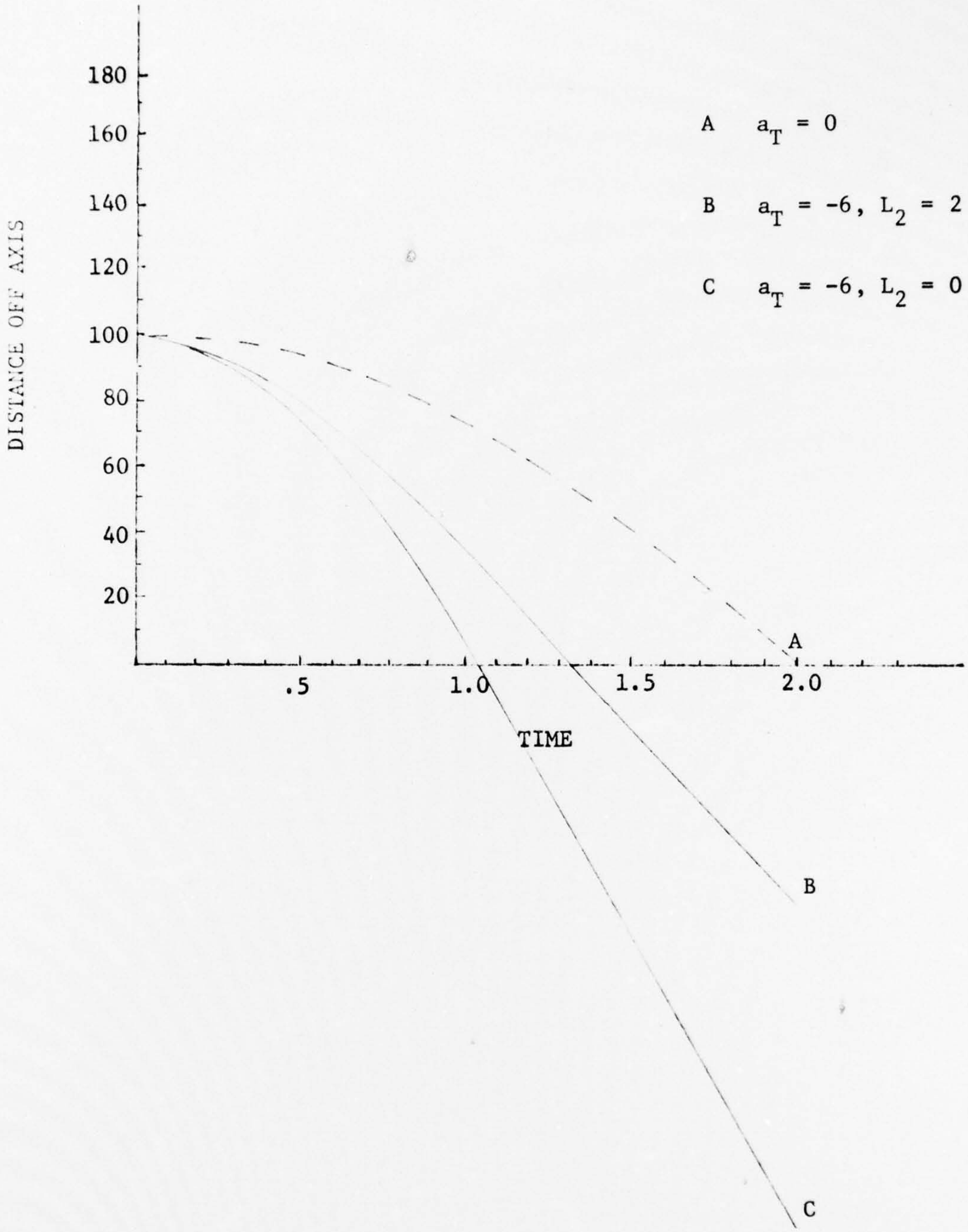


Figure 10. Miss Distance for One-Time Constant Missile.
Target Acceleration = -6g

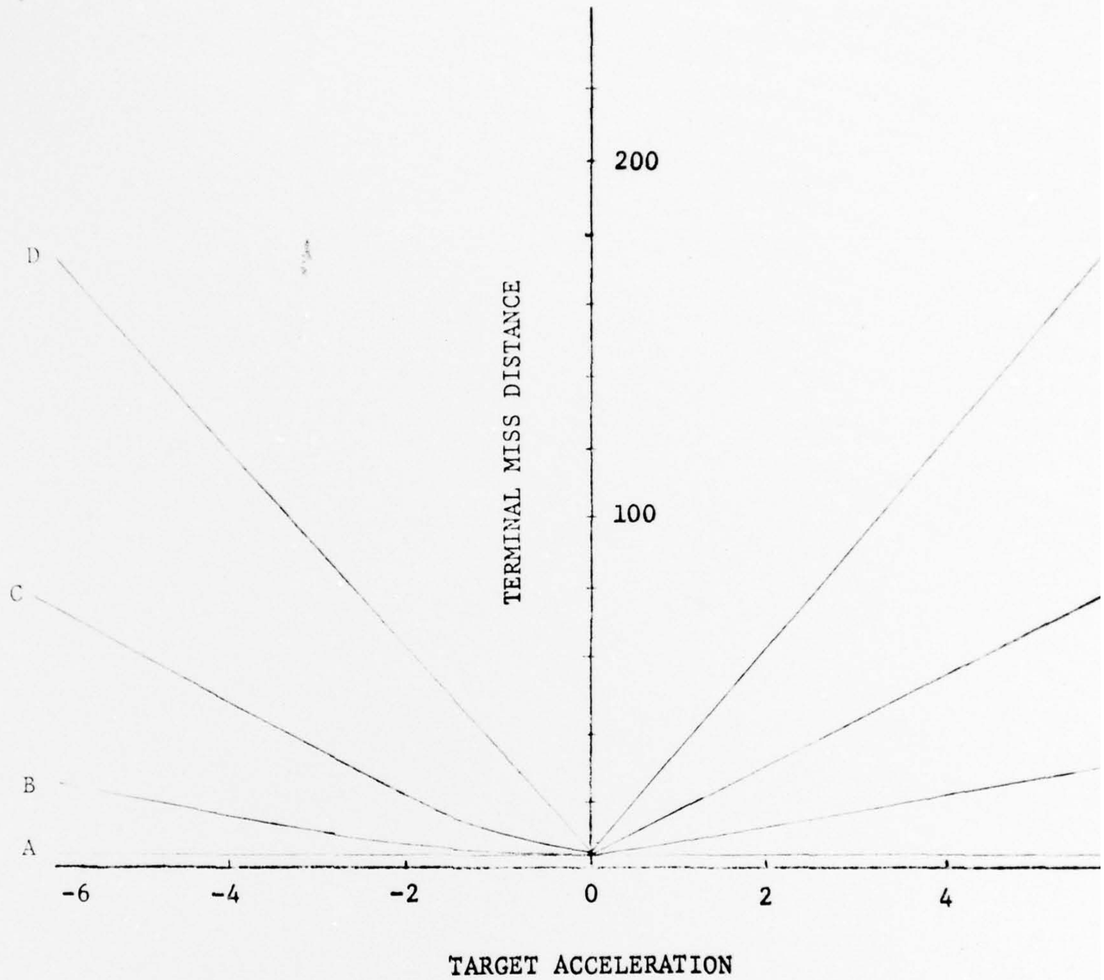


Figure 11. Terminal Miss Distance for One-Time Constant Missile

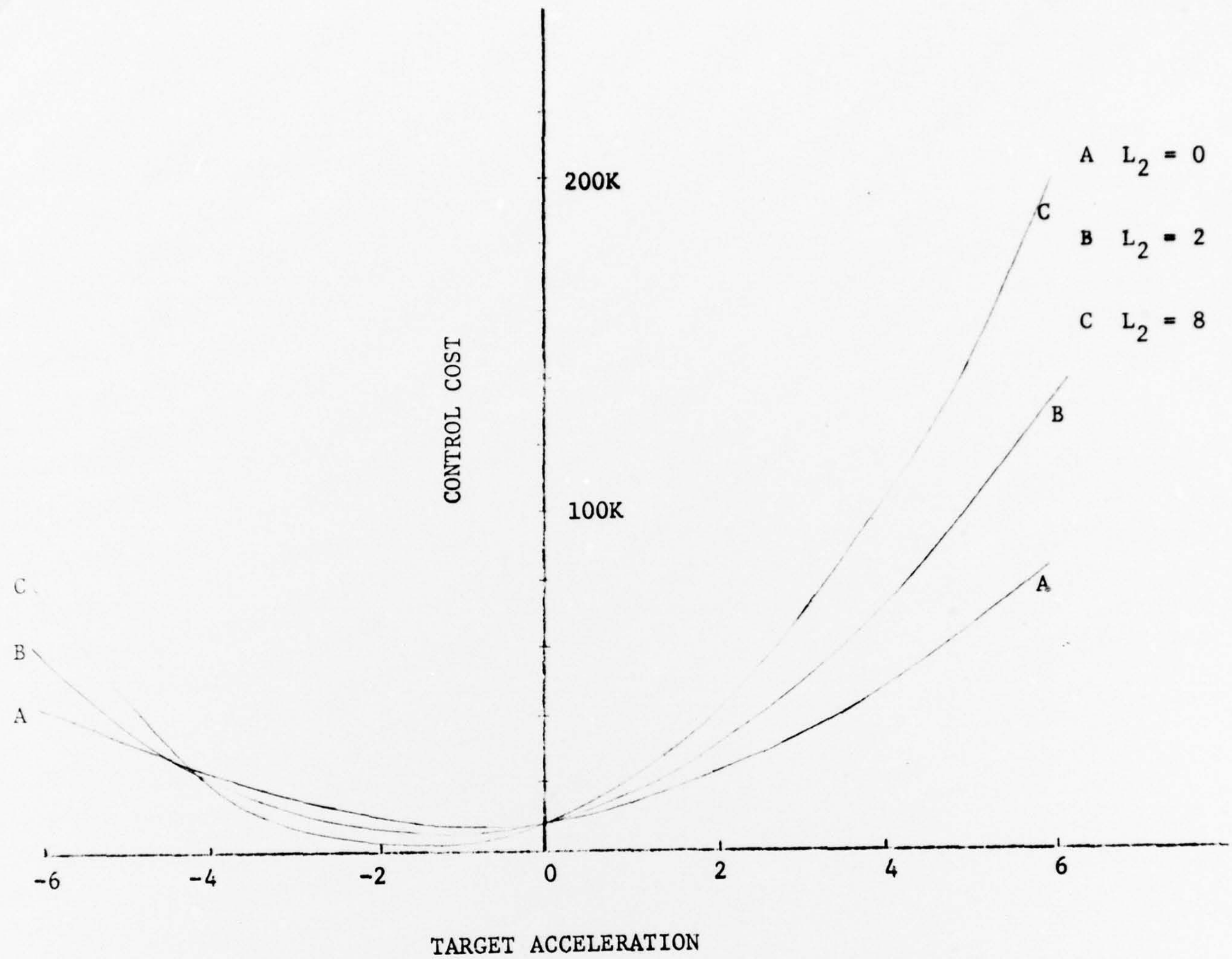


Figure 12. Control Cost for One-Time Constant Missile

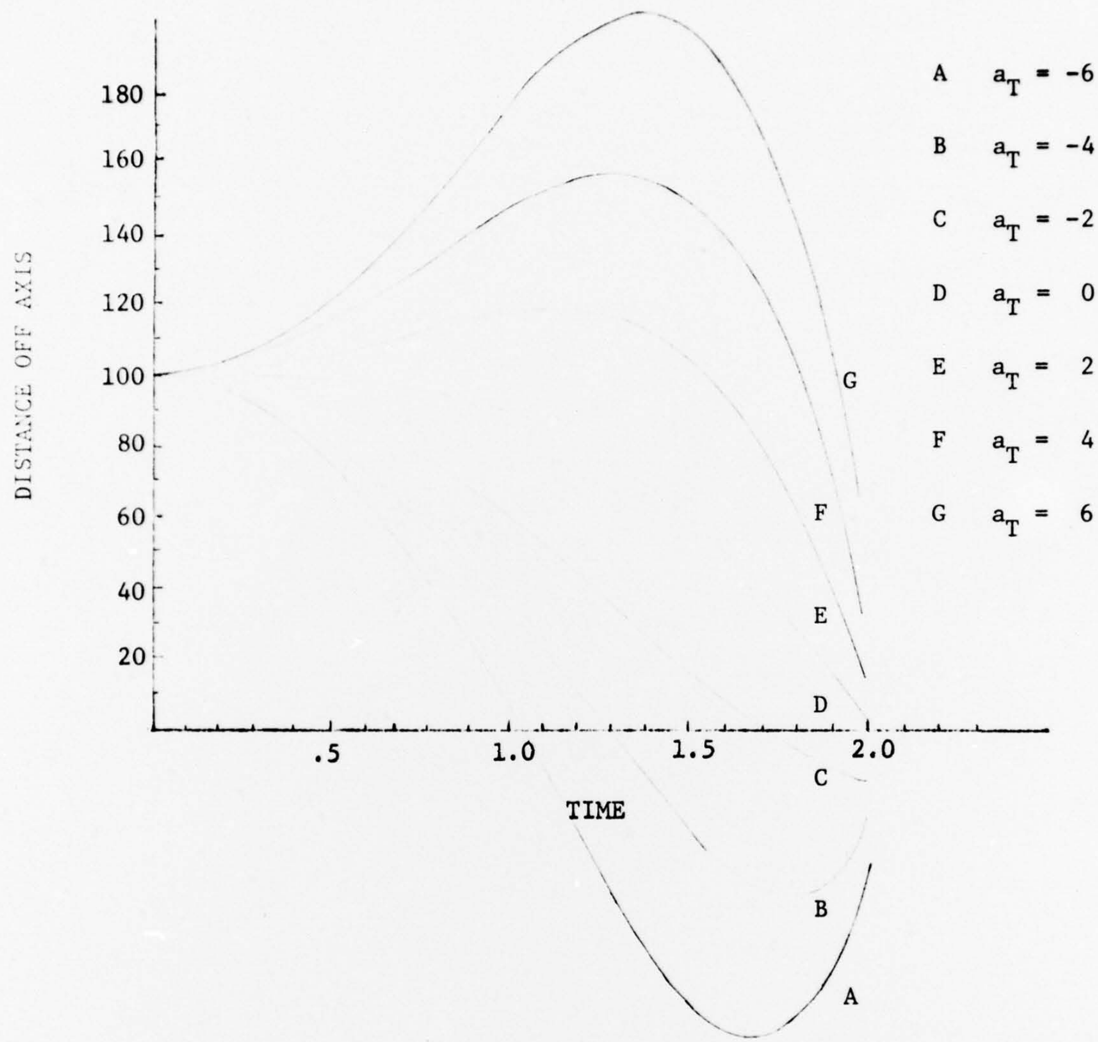


Figure 13. Miss Distance for Two-Time Constant Missile Without Target Estimation

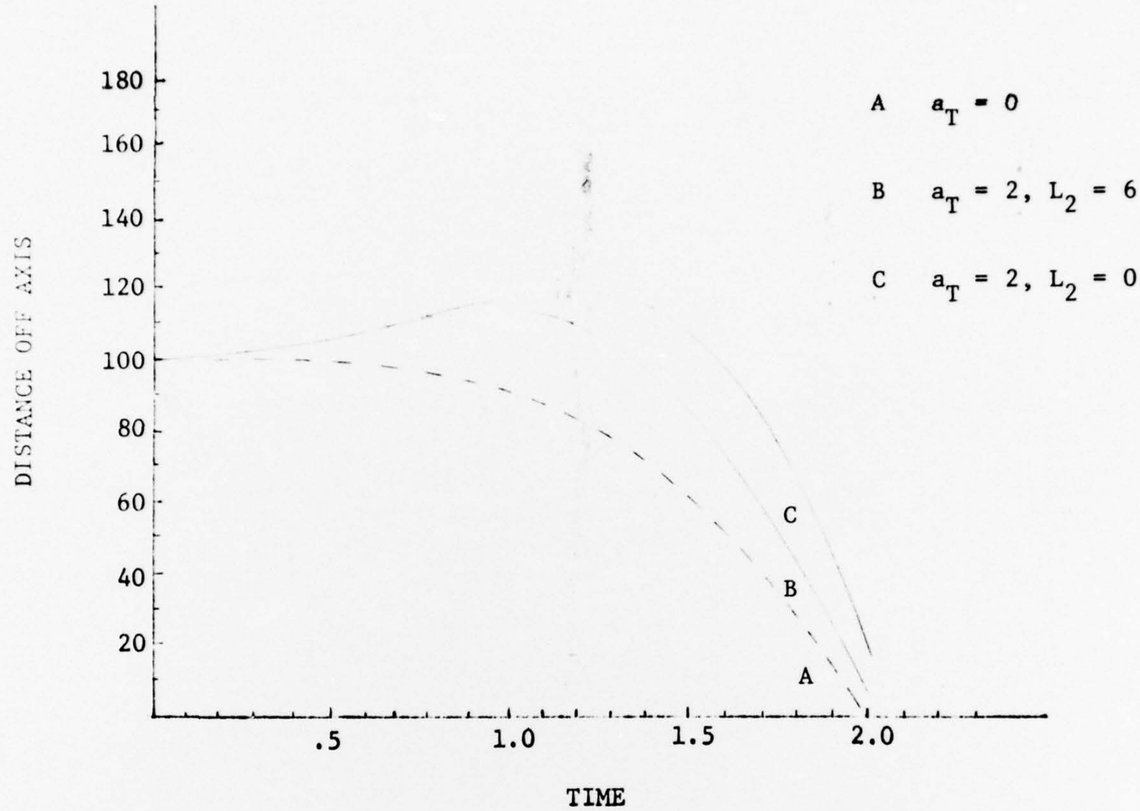


Figure 14. Miss Distance for Two-Time Constant Missile.
Target Acceleration = 2g

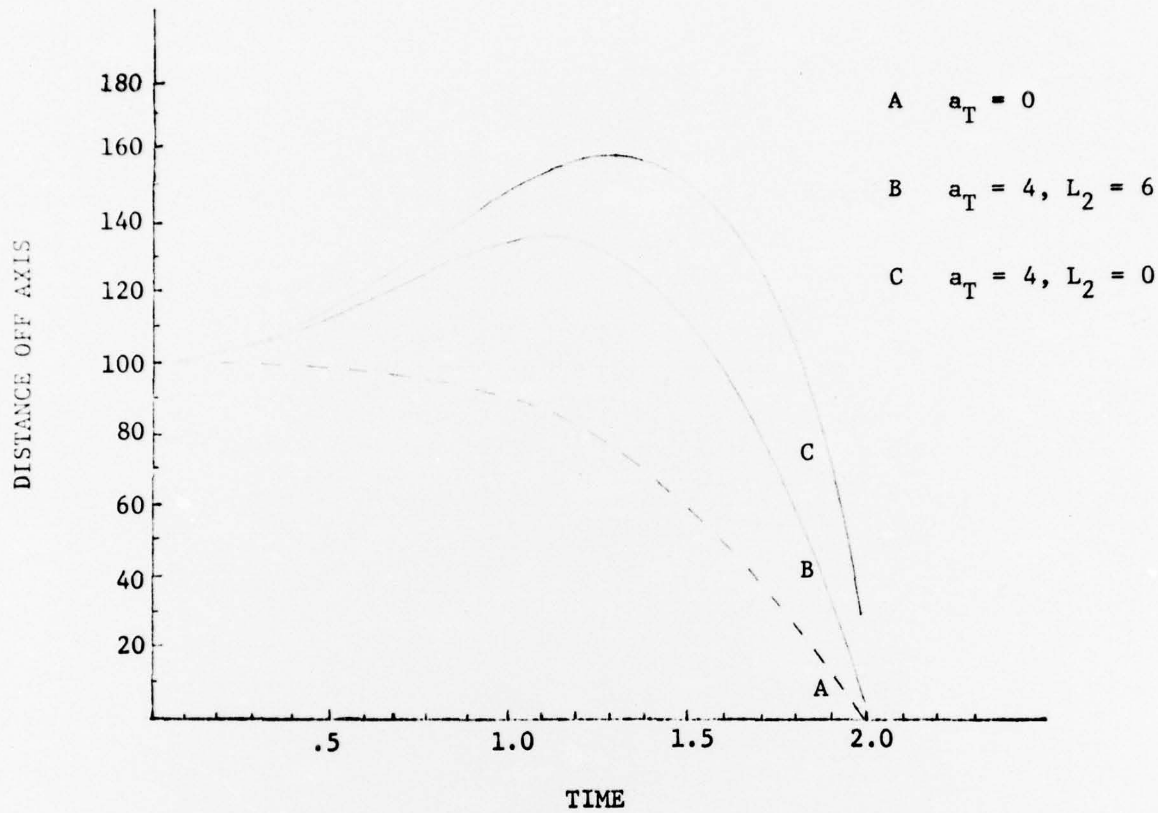


Figure 15. Miss Distance for Two-Time Constant Missile.
Target Acceleration = 4g

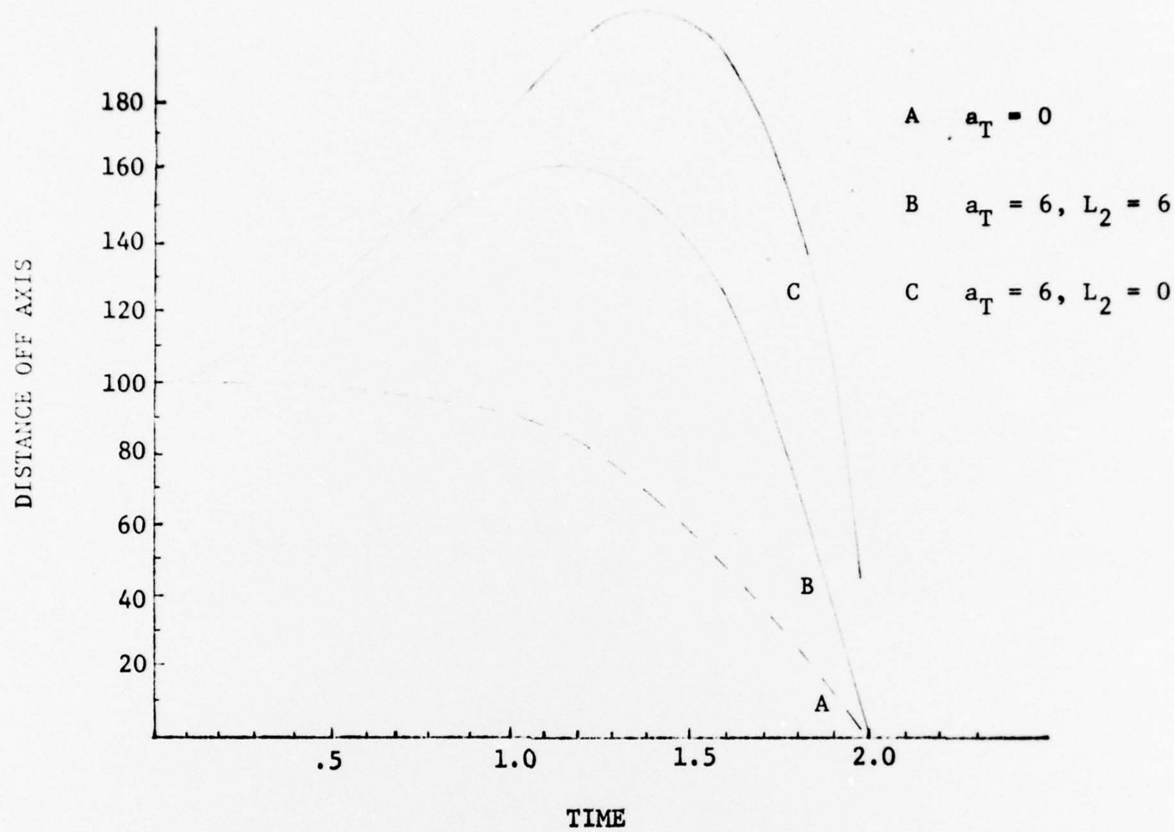


Figure 16. Miss Distance for Two-Time Constant Missile.
Target Acceleration = 6g

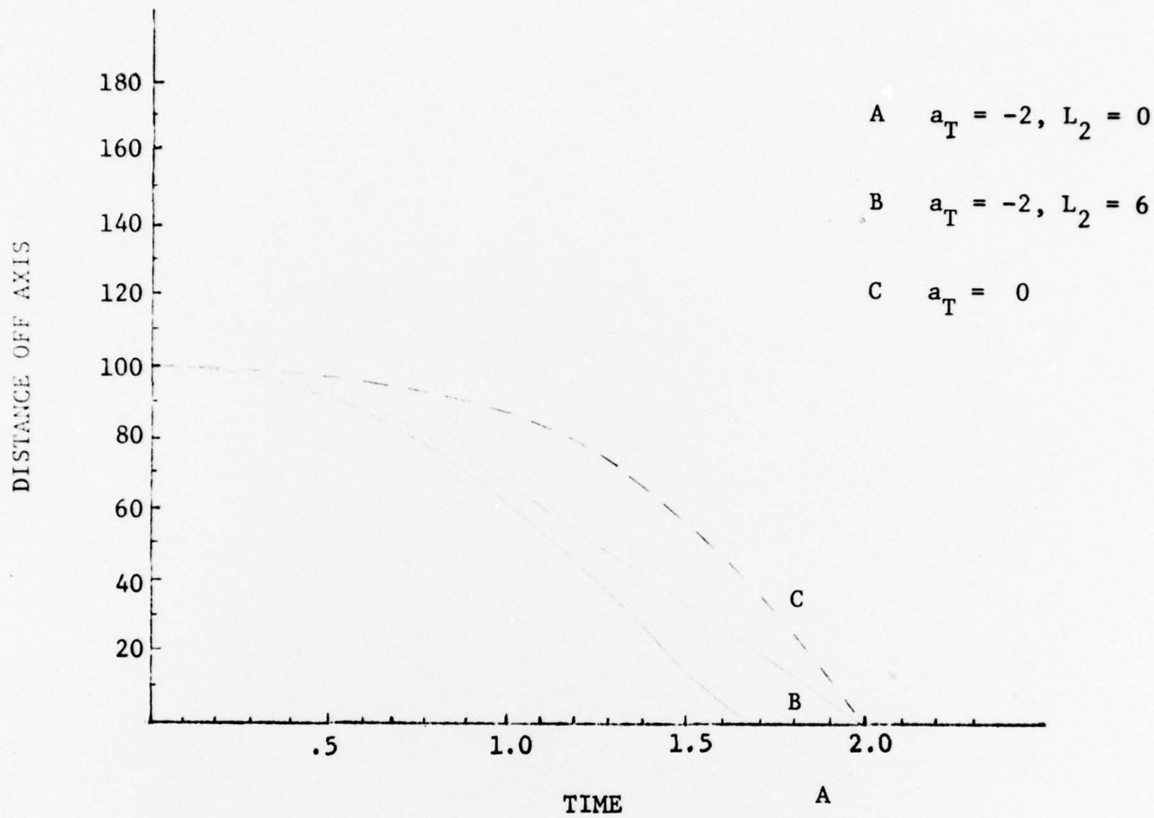


Figure 17. Miss Distance for Two-Time Constant Missile Target Acceleration = -2g

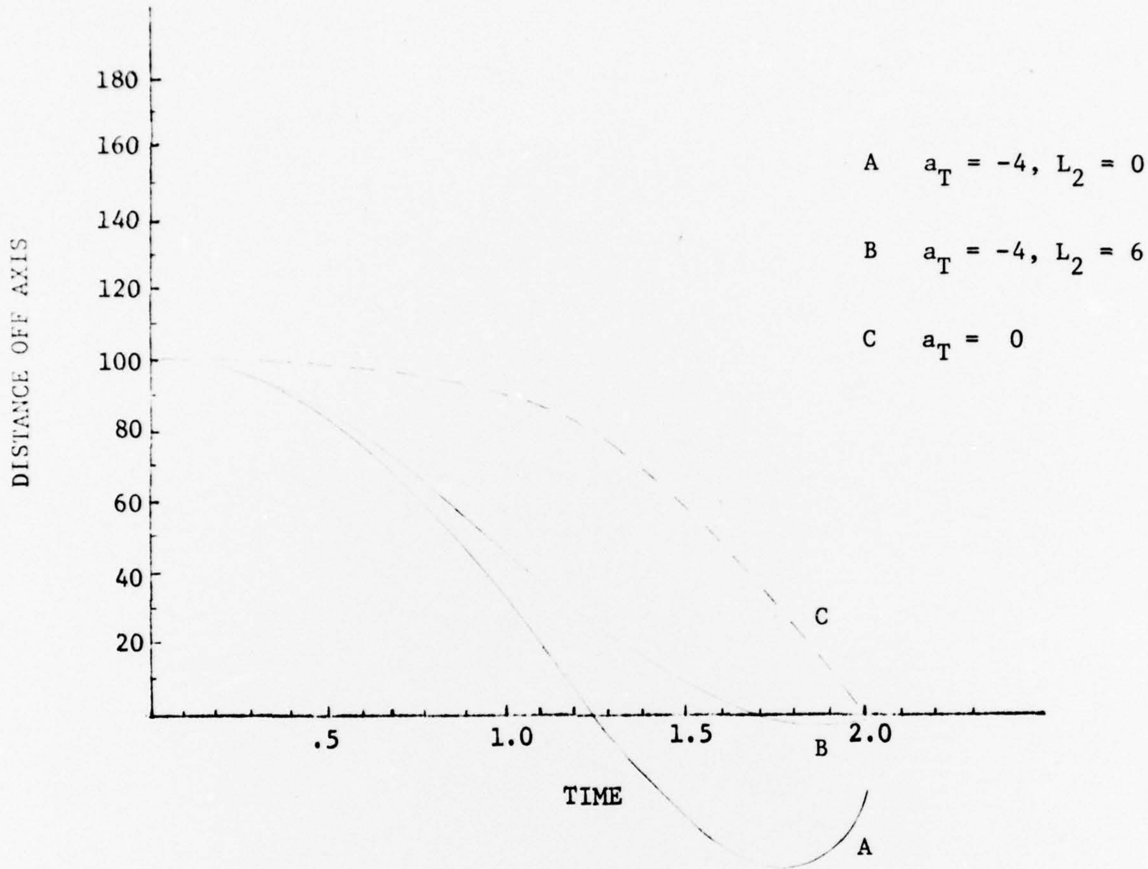


Figure 18. Miss Distance for Two-Time Constant Missile.
Target Acceleration = -4g

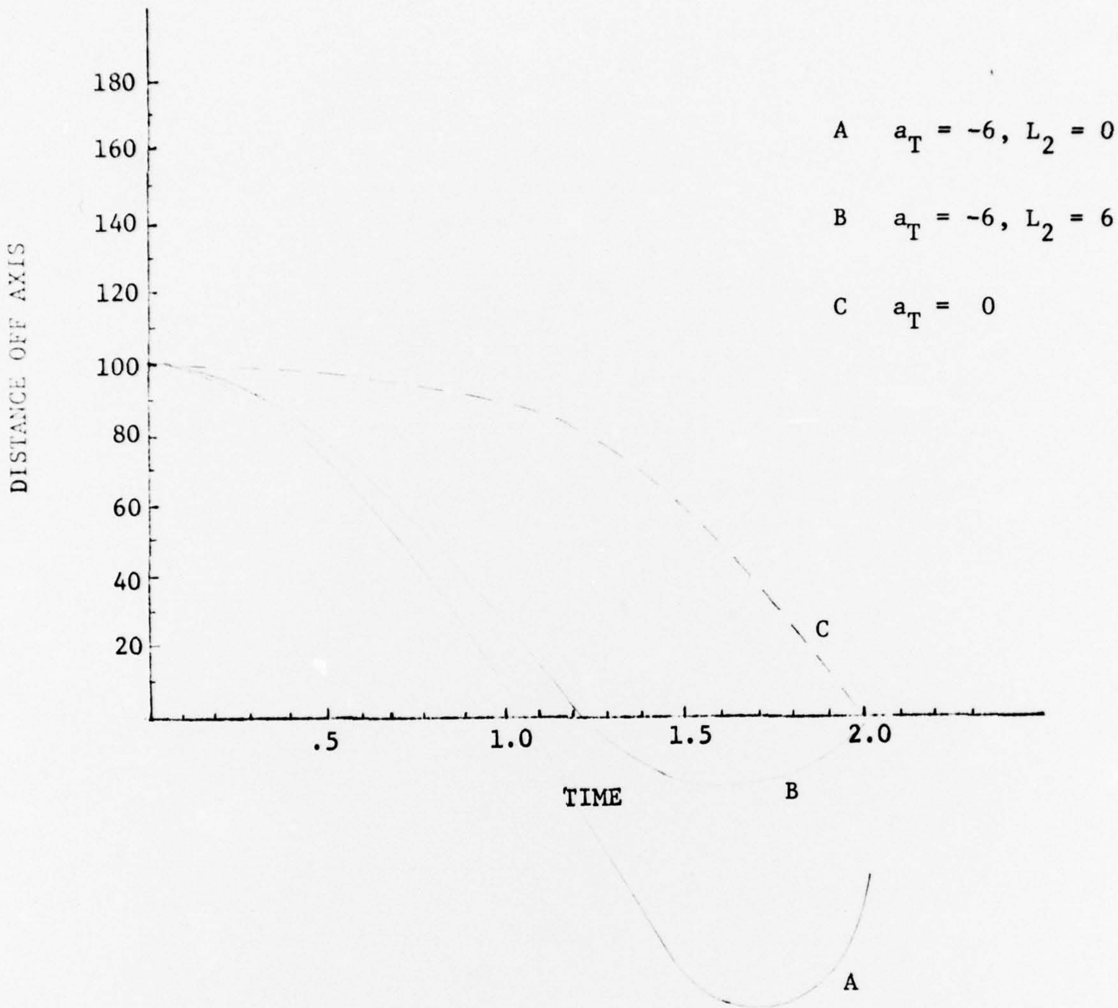


Figure 19. Miss Distance for Two-Time Constant Missile.
Target Acceleration = -6g

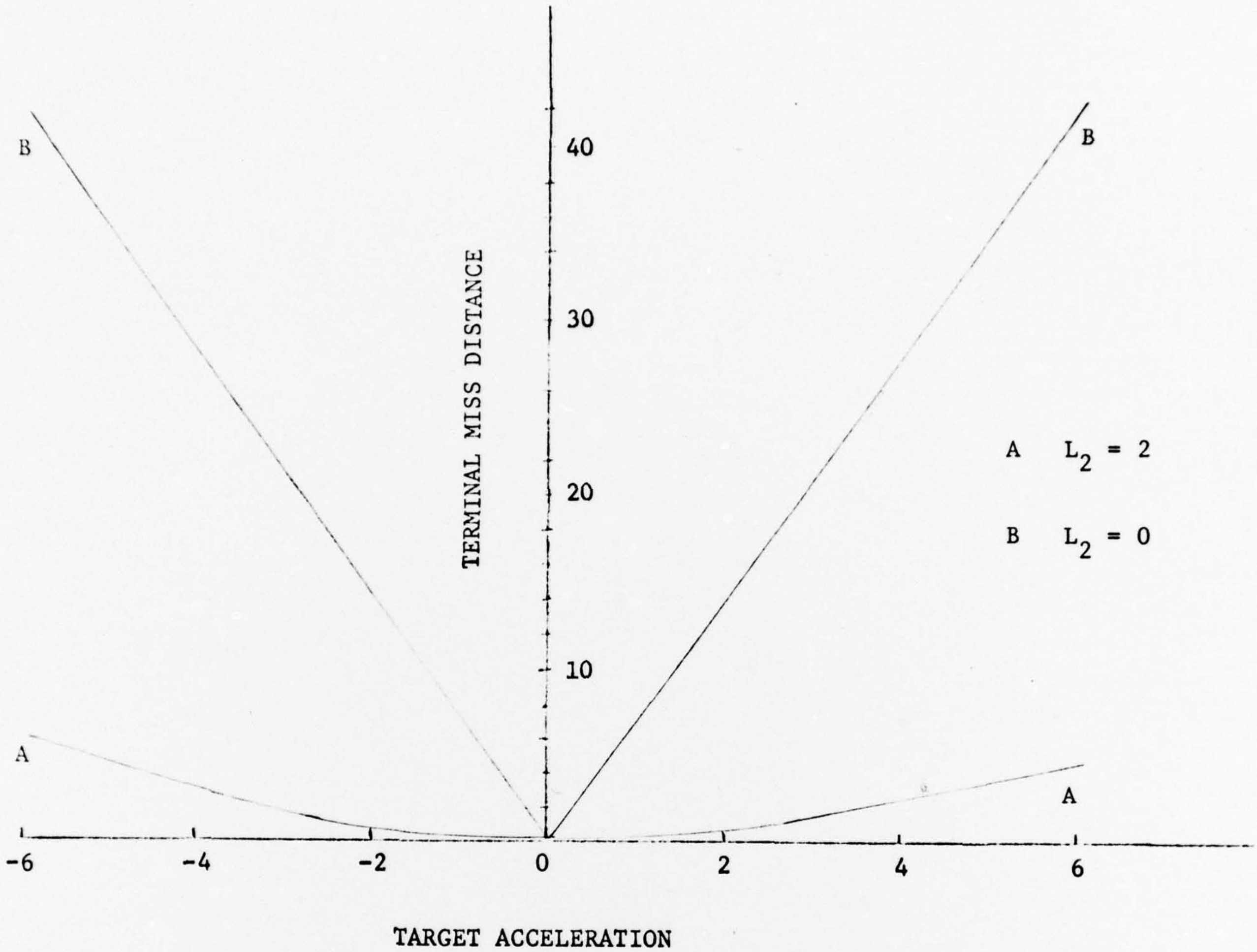


Figure 20. Terminal Miss Distance for Two-Time Constant Missile

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